

Generalized Hyperspaces and Non-Metrizable Fractals PREPRINT

Annie Carter
Daniel Lithio
Tristan Tager

November 18, 2010

Abstract

Much of the structure in metric spaces that allows for the creation of fractals exists in more generalized non-metrizable spaces. In particular the same theorems regarding the behavior of compact sets can be proven in the more general framework of β -spaces. However in most β -spaces, a set being compact (and more generally being totally bounded) is so restrictive as to render all fractal examples completely uninteresting. In this paper we provide a generalization of compact sets, continuous functions, and all the related machinery necessary for fractals to be defined as the unique fixed set of an IFS. We conclude by discussing some interesting examples of non-metrizable fractals.

1 Fractal Mechanics

Definition 1.1 (r-Open) A set U is called r -open if, for any $x \in U$, there is an $s =_L r$ such that $\beta(x, s) \subseteq U$.

Definition 1.2 A β -space is called swing complete if it satisfies that every $r \in R$ has a swing value $s \in R$ with $s =_L r$. Such a swing value is called a level swing value.

Definition 1.3 Given $r \in R$, a level swing sequence for r is a swing sequence $(r_i)_{i=1}^\infty$ for r satisfying that $r_i =_L r$ for all i .

Definition 1.4 (Roll Sets) Given a set A , a value $r \in R$, and a collection $(r_i)_{i=1}^\infty \subseteq R$, define

1. the roll of radius r about A , or the r -roll about A , to be the set

$$R_\beta(A, r) = \bigcup_{x \in A} \beta(x, r)$$

2. the iterated $(r_i)_{i=1}^\infty$ -roll about A as follows. We set $R_\beta^1(A, (r_i)_{i=1}^\infty) = R_\beta(A, r_1)$, and inductively set

$$R_\beta^{n+1}(A, (r_i)_{i=1}^\infty) = R_\beta(R_\beta^n(A, (r_i)_{i=1}^\infty), r_{n+1})$$

3. the sum r -roll about A to be the set

$$R_\Sigma(A, r) = \bigcup_{\substack{(r_i)_{i=1}^\infty \text{ a level} \\ \text{swing sequence of } r}} \bigcup_{n=1}^\infty R_\beta^n(A, (r_i)_{i=1}^\infty)$$

Definition 1.5 (Roll-Symmetric) A space (X, R, β) is called roll-symmetric if

$$R_\beta(R_\beta(A, r), s) = R_\beta(R_\beta(A, s), r)$$

for all $A \subseteq X$ and all $r, s \in R$.

Proposition 1.6 If (X, R, β) is roll-symmetric, then (X, R, β) is sum-roll-symmetric. That is,

$$R_\Sigma(R_\Sigma(A, r), s) = R_\Sigma(R_\Sigma(A, s), r)$$

for all $A \subseteq X$ and all $r, s \in R$.

Proof:

Let $x \in R_\Sigma(R_\Sigma(A, r), s)$. Then there are level swing sequences $(r_i)_{i=1}^\infty$ and $(s_i)_{i=1}^\infty$ and integers $n, m \in \mathbb{N}$ such that

$$\begin{aligned} x \in R_\beta^n(R_\beta^m(A, (r_i)_{i=1}^\infty), (s_i)_{i=1}^\infty) &= R_\beta^m(R_\beta^n(A, (s_i)_{i=1}^\infty), (r_i)_{i=1}^\infty) \\ &\subseteq R_\Sigma(R_\Sigma(A, s), r) \end{aligned}$$

Therefore

$$R_\Sigma(R_\Sigma(A, r), s) \subseteq R_\Sigma(R_\Sigma(A, s), r)$$

Since the reverse inclusion follows by symmetry, we have equality as desired. ■

The significance of these roll sets will become clear, as intuitively they provide us a way to generate “open balls” around subsets of our space X . The iterated $(r_i)_{i=1}^\infty$ -roll sets exist simply to define the sum r -roll sets, and the purpose of the sum r -roll sets is to generate r -open balls around our subsets of X .

Proposition 1.7 $R_\Sigma(A, r)$ is r -open for all $A \subseteq X$ and all $r \in R$.

Proof:

If $R_\Sigma(A, r)$ is empty (which occurs either when A is empty or when there is no level swing sequence for r in R), the statement is vacuously true. Supposing that there exists an $x \in R_\Sigma(A, r)$, there must be a level swing sequence $(r_i)_{i=1}^\infty$ for r such that

$$x \in \bigcup_{n=1}^{\infty} R_\beta^n(A, (r_i)_{i=1}^\infty)$$

Therefore, there is a k such that $x \in R_\beta^k(A, (r_i)_{i=1}^\infty)$. This implies that

$$\beta(x, r_{k+1}) \subseteq R_\beta^{k+1}(A, (r_i)_{i=1}^\infty) \subseteq \bigcup_{n=1}^{\infty} R_\beta^n(A, (r_i)_{i=1}^\infty) \subseteq R_\Sigma(A, r)$$

But since $(r_i)_{i=1}^\infty$ is a level swing sequence, $r_{k+1} =_L r$, and since x was arbitrary, $R_\Sigma(A, r)$ must be r -open. ■

Definition 1.8 (Level-Unbounded) A β -space (X, R, β) is called level-unbounded if for every $r \in R$ there exists an $s \in R$ with $s <_L r$.

The reader will note that when we focus on level-unbounded spaces, we fail to obtain a generalization of the theorems we have on metric spaces since metric spaces are not level-unbounded. However, in the case that a space fails to be level-unbounded, there will be a smallest level, generated say by $r \in R$. Then we echo more exactly the standard metric space construction, and form the hyperspace of r -compact sets (which end up being compact), and show that we get a β -structure in that fashion. Since we wish to generate fractal behavior for wholly new sorts of spaces, we will bypass this construction altogether. Because of this, and because of the additional structure we obtain, we will focus on level-unbounded spaces for the rest of this section.

Definition 1.9 Let (X, R, β) be a β -space and let $\mathcal{X} \subseteq \wp(X)$ be a collection of subsets. For any $A \in \mathcal{X}$ and $r \in R$, define

$$\beta_\mathbb{H}(A, r) = \{B \in \mathcal{X} : B \subseteq R_\Sigma(A, r) \text{ and } A \subseteq R_\Sigma(B, r)\}$$

Proposition 1.10 Let (X, R, β) be an ordered roll-symmetric β -space and let $\mathcal{X} = \wp(X)$. For any $A \in \mathcal{X}$ and any $r \in R$, if $B \in \mathcal{X}$ satisfies $B \in \beta_\mathbb{H}(A, r)$, for any $t <_L r$, $\beta_\mathbb{H}(B, t) \subseteq \beta_\mathbb{H}(A, r)$.

Proof:

Since $B \in \beta_{\mathbb{H}}(A, r)$, for any $C \in \beta_{\mathbb{H}}(B, t)$, we know that

1. $B \subseteq R_{\Sigma}(A, r)$
2. $A \subseteq R_{\Sigma}(B, r)$
3. $C \subseteq R_{\Sigma}(B, t)$
4. $B \subseteq R_{\Sigma}(C, t)$

Now clearly $R_{\Sigma}(A, r)$ an r -open set implies that for any $x \in R_{\Sigma}(A, r)$, $\beta(x, t) \subseteq R_{\Sigma}(A, r)$. From this and (1) it follows that $R_{\Sigma}(B, t) \subseteq R_{\Sigma}(A, r)$. From (3) this implies that

$$C \subseteq R_{\Sigma}(A, r)$$

Now from (2) and (4) we have that

$$A \subseteq R_{\Sigma}(B, r) \subseteq R_{\Sigma}(R_{\Sigma}(C, t), r)$$

But by Proposition 1.6, since (X, R, β) is roll-symmetric, we have that

$$A \subseteq R_{\Sigma}(R_{\Sigma}(C, r), t)$$

But again, since $R_{\Sigma}(C, r)$ is r -open, it follows that

$$A \subseteq R_{\Sigma}(R_{\Sigma}(C, r), t) = R_{\Sigma}(C, r)$$

which establishes that $C \in \beta_{\mathbb{H}}(A, r)$, so that

$$\beta_{\mathbb{H}}(B, t) \subseteq \beta_{\mathbb{H}}(A, r)$$

as desired. \blacksquare

Lemma 1.11 *Let (X, R, β) be symmetric and roll-symmetric. Then whenever s is a swing value for r , $R_{\Sigma}(R_{\Sigma}(A, s), s) \subseteq R_{\Sigma}(A, r)$.*

Proof:

First we note that if our space is symmetric, then whenever $y \in \beta(x, s)$, $x \in \beta(y, s)$ so that

$$\beta(x, s), \beta(y, s) \subseteq \beta(x, r)$$

Clearly this implies that $R_{\beta}(R_{\beta}(A, s), s) \subseteq R_{\beta}(A, r)$. Now let $(s_i)_{i=1}^{\infty}, (t_i)_{i=1}^{\infty}$ be swing sequences for s , and set

$$(r_i)_{i=1}^{\infty} = (r, \max\{s_1, t_1\}, \max\{s_2, t_2\}, \dots)$$

Clearly $(r_i)_{i=1}^{\infty}$ is a swing sequence for r , and s_i, t_i are swing values for r_i for all i . We will now show that

$$R_{\beta}^n(R_{\beta}^n(A, (s_i)_{i=1}^{\infty}), (t_i)_{i=1}^{\infty}) \subseteq R_{\beta}^n(A, (r_i)_{i=1}^{\infty})$$

for all n . Noting that we have already shown the base case for this induction, suppose it is true for n . Then

$$\begin{aligned}
R_\beta^{n+1}(R_\beta^{n+1}(A, (s_i)_{i=1}^\infty), (t_i)_{i=1}^\infty) &= R_\beta(R_\beta^n(R_\beta(R_\beta^n(A, (s_i)_{i=1}^\infty), s_{n+1}), (t_i)_{i=1}^\infty), t_{n+1}) \\
&= R_\beta(R_\beta(R_\beta^n(R_\beta^n(A, (s_i)_{i=1}^\infty), (t_i)_{i=1}^\infty), s_{n+1}), t_{n+1}) \\
&\subseteq R_\beta(R_\beta(R_\beta^n(A, (r_i)_{i=1}^\infty), s_{n+1}), t_{n+1}) \\
&\subseteq R_\beta(R_\beta(R_\beta^n(A, (r_i)_{i=1}^\infty), r_{n+2}), r_{n+2}) \\
&\subseteq R_\beta(R_\beta^n(A, (r_i)_{i=1}^\infty), r_{n+1}) \\
&= R_\beta^{n+1}(A, (r_i)_{i=1}^\infty)
\end{aligned}$$

as desired. Finally, since $(s_i)_{i=1}^\infty, (t_i)_{i=1}^\infty$ were arbitrary, we have that

$$R_\Sigma(R_\Sigma(A, s), s) \subseteq R_\Sigma(A, r)$$

which completes the proof. ■

Lemma 1.12 *Suppose that $r_1, r_2, r_3 \in R$ satisfy $r_i =_L r_j$ and r_{i+1} a swing value for r_i . Then $R_\Sigma(A, r_3) \subseteq R_\beta(A, r)$ for any $A \subseteq X$.*

Proof:

Let $x \in R_\Sigma(A, r_3)$. Then there is a level swing sequence $(s_i)_{i=1}^\infty$ for r_3 and an integer $n \in \mathbb{N}$ such that

$$x \in R_\beta^n(A, (s_i)_{i=1}^\infty)$$

In particular this implies that there is a collection $(a_i)_{i=1}^n$ such that $a_1 \in A$, $x \in \beta(a_n, s_n)$, and in general $a_{i+1} \in \beta(a_i, s_i)$. Then by the Geometric Series Lemma, $\beta(a_i, s_i) \subseteq \beta(a_1, r)$ for all i . Thus

$$x \in \beta(a_n, s_n) \subseteq \beta(a_1, r) \subseteq R_\beta(A, r)$$

which completes the proof. ■

Lemma 1.13 *Let (X, R, β) be an ordered space. Then whenever $s \leq r$, $R_\Sigma(A, s) \subseteq R_\Sigma(A, r)$ for all $A \subseteq X$.*

Proof:

The statement follows trivially if $R_\Sigma(A, s)$ is empty. Suppose there is an $x \in R_\Sigma(A, s)$. Then there is a swing sequence $(s_i)_{i=1}^\infty$ for s such that

$$x \in \bigcup_{n=1}^\infty R_\beta^n(A, (s_i)_{i=1}^\infty)$$

Now for any y , let $z \in \beta(y, s_2)$. Then $\beta(y, s_2) \subseteq \beta(z, s) \subseteq \beta(z, r)$, so that s_2 is a swing value of r . Therefore it follows that the sequence $(t_i)_{i=1}^\infty = (r, s_2, s_3, \dots)$ is a swing sequence for r . But clearly

$$x \in \bigcup_{n=1}^\infty R_\beta^n(A, (t_i)_{i=1}^\infty) \subseteq R_\Sigma(A, r)$$

which completes the proof. ■

Theorem 1.14 *Let (X, R, β) be ordered, symmetric, roll-symmetric, level-unbounded, and swing-complete, and let $\mathcal{X} \subseteq \wp(X)$ be a collection of nonempty subsets. Then the space $(\mathcal{X}, R, \beta_{\mathbb{H}})$ is an ordered β -space. Further, if each $A \in \mathcal{X}$ is closed, then \mathcal{X} is a Hausdorff β -space.*

Proof:

First we verify the four conditions of β -spaces.

1. $A \in \beta_{\mathbb{H}}(A, r)$. Since our space is swing-complete, there exists a level swing sequence $(r_i)_{i=1}^{\infty}$ for r , so trivially this condition is satisfied.
2. $\beta_{\mathbb{H}}(A, r)$ is open. We take the topology on \mathcal{X} to be induced by the $\beta_{\mathbb{H}}$ balls, so this is true by definition of the topology.
3. To show this condition, it suffices to show that when $C \in \beta_{\mathbb{H}}(A, r) \cap \beta_{\mathbb{H}}(B, s)$, there is a t such that

$$\beta_{\mathbb{H}}(C, t) \subseteq \beta_{\mathbb{H}}(A, r) \cap \beta_{\mathbb{H}}(B, s)$$

Since our space is level-ordered and level-unbounded, there is a $t \in R$ such that $t <_L r$ and $t <_L s$. Then by Proposition 1.10, $\beta_{\mathbb{H}}(C, t)$ is contained in the intersection as required.

4. Given $r \in R$, we claim that any $s \in R$ that is a swing value for r in the space (X, R, β) is also a swing value for r in the space $(\mathbb{H}(X), R, \beta_{\mathbb{H}})$. Suppose that $B \in \beta_{\mathbb{H}}(A, s)$, and let $C \in \beta_{\mathbb{H}}(A, s)$ be arbitrary. Then we have that

- (a) $B \subseteq R_{\Sigma}(A, s)$
- (b) $A \subseteq R_{\Sigma}(B, s)$
- (c) $C \subseteq R_{\Sigma}(A, s)$
- (d) $A \subseteq R_{\Sigma}(C, s)$

From (b) and (c) we obtain that

$$C \subseteq R_{\Sigma}(A, s) \subseteq R_{\Sigma}(R_{\Sigma}(B, s), s) \subseteq R_{\Sigma}(B, r)$$

where the last inclusion follows from Lemma 1.11. Similarly from (a) and (d) we get that

$$B \subseteq R_{\Sigma}(A, s) \subseteq R_{\Sigma}(R_{\Sigma}(C, s), s) \subseteq R_{\Sigma}(C, r)$$

which gives us that $C \in \beta_{\mathbb{H}}(B, r)$. But since C was arbitrary, we have that

$$\beta_{\mathbb{H}}(A, s) \subseteq \beta_{\mathbb{H}}(B, r)$$

as desired.

To see that the hyperspace is ordered, we recall that by Lemma 1.13, when $s \leq r$ in the ordering on (X, R, β) we have that $R_{\Sigma}(A, s) \subseteq R_{\Sigma}(A, r)$. Then if $B \in \beta_{\mathbb{H}}(A, s)$, we have

1. $B \subseteq R_{\Sigma}(A, s) \subseteq R_{\Sigma}(A, r)$
2. $A \subseteq R_{\Sigma}(B, s) \subseteq R_{\Sigma}(B, r)$

so that $B \in \beta_{\mathbb{H}}(A, r)$ as desired.

Finally, we show that if \mathcal{X} contains only closed sets, then our space is Hausdorff. Let $A, B \in \mathcal{X}$, and suppose that $A \neq B$. Since A and B are both closed, there must exist a point $x \in A \setminus B$ (without loss of generality) and an $r \in R$ such that $\beta(x, r) \cap B = \emptyset$. Let $(r_i)_{i=1}^{\infty}$ be a level swing sequence for r . Then we claim that

$$\beta_{\mathbb{H}}(A, r_5) \cap \beta_{\mathbb{H}}(B, r_5) = \emptyset$$

Suppose to the contrary that we had some $C \in \beta_{\mathbb{H}}(A, r_5) \cap \beta_{\mathbb{H}}(B, r_5)$. Then by definition of $\beta_{\mathbb{H}}$, this gives us

1. $C \subseteq R_{\Sigma}(A, r_5)$
2. $A \subseteq R_{\Sigma}(C, r_5)$
3. $C \subseteq R_{\Sigma}(B, r_5)$
4. $B \subseteq R_{\Sigma}(C, r_5)$

From (2) and (3) we have that

$$A \subseteq \mathbb{R}_{\Sigma}(R_{\Sigma}(B, r_5), r_5) \subseteq R_{\Sigma}(B, r_4)$$

By Lemma 1.12, $R_{\Sigma}(B, r_4) \subseteq R_{\beta}(B, r_2)$, so that since $x \in R_{\Sigma}(B, r_4)$, we know that there is a $y \in B$ such that $x \in \beta(y, r_2)$. Therefore $y \in \beta(x, r)$. But this contradicts that $\beta(x, r) \cap B = \emptyset$, so such a set C cannot exist. \blacksquare

Theorem 1.15 *Let (X, R, β) be an ordered β -space, and let $(\wp(X), R, \beta_{\mathbb{H}})$ be a hyperspace. Then if $f_1, f_2, \dots, f_n : X \rightarrow X$ are contractions on X with f_i having degree of contraction N_i , the function $F : \mathcal{X} \rightarrow \mathcal{X}$ given by*

$$F(A) = \bigcup_{i=1}^n f_i(A)$$

is also a contraction, with degree of contraction $M = \max\{N_1, \dots, N_n\}$.

Proof:

Since all the f_i are contractions, F satisfies

$$\begin{aligned}
F(R_\beta(A, r)) &= F\left(\bigcup_{x \in A} \beta(x, r)\right) \\
&= \bigcup_{x \in A} F(\beta(x, r)) \\
&= \bigcup_{x \in A} \bigcup_{i=1}^n f_i(\beta(x, r)) \\
&\subseteq \bigcup_{x \in A} \bigcup_{i=1}^n \beta(f_i(x), r) \\
&= \bigcup_{i=1}^n \bigcup_{x \in f_i(A)} \beta(x, r) \\
&= \bigcup_{x \in F(A)} \beta(x, r) \\
&= R_\beta(F(A), r)
\end{aligned}$$

From this it follows that $F(R_\Sigma(A, r)) \subseteq R_\Sigma(F(A), r)$. Now we want to show that $F(\beta_\mathbb{H}(A, r)) \subseteq \beta_\mathbb{H}(F(A), r)$. Let $B \in \beta_H(A, r)$. Then we have that

1. $B \subseteq R_\Sigma(A, r)$
2. $A \subseteq R_\Sigma(B, r)$

From this we have that

1. $F(B) \subseteq F(R_\Sigma(A, r)) \subseteq R_\Sigma(F(A), r)$
2. $F(A) \subseteq F(R_\Sigma(B, r)) \subseteq R_\Sigma(F(B), r)$

which gives that $F(B) \in \beta_\mathbb{H}(F(A), r)$ as desired. Now proceeding similarly, clearly we can see

$$\begin{aligned}
F^M(R_\beta(A, r)) &= F^M\left(\bigcup_{x \in A} \beta(x, r)\right) \\
&= \bigcup_{x \in A} F^M(\beta(x, r)) \\
&= \bigcup_{x \in A} \bigcup_{i=1}^n f_i^M(\beta(x, r)) \\
&= \bigcup_{i=1}^n \bigcup_{x \in A} f_i^M(\beta(x, r)) \\
&\subseteq \bigcup_{i=1}^n \bigcup_{x \in f_i^M(A)} \beta(x, s) \\
&= \bigcup_{x \in F^M(A)} \beta(x, s) \\
&= R_\beta(F^M(A), s)
\end{aligned}$$

From this it follows that $F^M(R_\Sigma(A, r)) \subseteq R_\Sigma(F^M(A), s)$, and by the same argument as before, this gives that $F^M(\beta_\mathbb{H}(A, r)) \subseteq \beta_\mathbb{H}(F^M(A), s)$ as desired. ■

As usual, we call the collection of contractions $(f_i)_{i=1}^n$ an *iterated function system*, and will often use the same term to refer to the induced function F on the hyperspace.

Definition 1.16 (Level-Countable) *A space is called level-countable if whenever $r =_L s$ and $(r_i)_{i=1}^\infty$ is a swing sequence for r , there is a k such that $r_k \leq s$.*

Lemma 1.17 *Let (X, R, β) be an ordered level-countable swing-complete β -space, and let $(\mathbb{H}(X), R, \beta_\mathbb{H})$ be a hyperspace. Then*

1. $s <_L r$ in X implies $s \leq_L r$ in $\mathbb{H}(X)$
2. $s =_L r$ in X implies that $s =_L r$ in $\mathbb{H}(X)$
3. If $\mathbb{H}(X)$ contains all singleton sets, then $s <_L r$ in X implies $s <_L r$ in $\mathbb{H}(X)$

Proof:

1. By Lemma ??, if $s <_L r$ we know that $s < r$. We note that

$$L_\mathbb{H}(A, s) = \{B : \exists \text{ swing sequence } (s_i)_{i=1}^\infty \text{ such that } B \in \beta_\mathbb{H}(A, s_i) \forall i\}$$

Then for any $B \in L_\mathbb{H}(A, s)$, with $(s_i)_{i=1}^\infty$ the relevant swing sequence, we note that because $s < r$, the sequence $(r_i)_{i=1}^\infty = (r, s_2, s_3, \dots)$ is a swing sequence for r . Then we have that

$$B \subseteq R_\Sigma(A, s_i) \subseteq R_\Sigma(A, r_i)$$

where the last inclusion is trivial for $i \geq 2$, and follows for $i = 1$ by Lemma 1.13. The inclusion $A \subseteq R_\Sigma(B, r_i)$ follows by the identical argument, and so we have that $B \in L_\mathbb{H}(A, r)$. Thus $L_\mathbb{H}(A, s) \subseteq L_\mathbb{H}(A, r)$. But since A was arbitrary, this implies that $s \leq_L r$ in $\mathbb{H}(X)$.

2. Now suppose that $s =_L r$ in X . Without loss of generality, suppose that $s \leq r$ in X . Then by the above argument, we have that $s \leq_L r$ in $\mathbb{H}(X)$. Now since our space is level-countable and $s =_L r$ in X , by swing-completeness there is a level swing sequence $(r_i)_{i=1}^\infty$ and a $k \in \mathbb{N}$ such that $r_k \leq s$. This gives us that

$$L_\mathbb{H}(A, r_k) \subseteq L_\mathbb{H}(A, s) \subseteq L_\mathbb{H}(A, r).$$

To complete the proof, therefore, it suffices to show that $L_\mathbb{H}(A, r) \subseteq L_\mathbb{H}(A, r_k)$. Let $B \in L_\mathbb{H}(A, r)$, and let $(t_i)_{i=1}^\infty$ be a swing sequence for r such that $B \in \beta_\mathbb{H}(A, t_i)$ for all i . Without loss of generality we can take $(t_i)_{i=1}^\infty$ to be a level swing sequence. Then again since our space is level-countable and since $t_1 = r =_L r_k$, there is an $n \in \mathbb{N}$ such that $t_n \leq r_k$. This implies that the sequence

$$(v_i)_{i=1}^\infty = (r_k, t_{n+1}, t_{n+2}, \dots)$$

is a swing sequence for r_k . But clearly $B \in \beta_{\mathbb{H}}(A, v_i)$ for all i , so that $B \in L_{\mathbb{H}}(A, r_k)$. Since B was arbitrary, this gives us that $L_{\mathbb{H}}(A, r) \subseteq L_{\mathbb{H}}(A, r_k)$ as desired.

3. Suppose that $s <_L r$ in X and that $\mathbb{H}(X)$ contains all singleton sets. Then there are $x, y \in X$ such that $y \in L(x, r) \setminus L(x, s)$. Since part (1) gives us that $s \leq_L r$ in $\mathbb{H}(X)$, to show that $s <_L r$ in $\mathbb{H}(X)$ we will show that $\{y\} \in L_{\mathbb{H}}(\{x\}, r) \setminus L_{\mathbb{H}}(\{x\}, s)$. Since $y \in L(x, r)$, there is a swing sequence $(r_i)_{i=1}^{\infty}$ for r such that, for all i ,

$$y \in \beta(x, r_i) = R_{\beta}(\{x\}, r_i) \subseteq R_{\Sigma}(\{x\}, r_i)$$

Now for any i , since $y \in \beta(x, r_{i+1})$, necessarily $x \in \beta(y, r_i)$, so that the above equality establishes that $x \in R_{\Sigma}(\{y\}, r_i)$. Therefore we have that $\{y\} \in \beta_{\mathbb{H}}(\{x\}, r_i)$ for all i , so that $\{y\} \in L_{\mathbb{H}}(\{x\}, r)$ as desired. Now suppose to the contrary that $\{y\} \in L_{\mathbb{H}}(\{x\}, s)$. Then there is a swing sequence $(s_i)_{i=1}^{\infty}$ for s such that $\{y\} \in \beta_{\mathbb{H}}(\{x\}, s_i)$ for all i . Since $y \notin L(x, s)$, there must exist a k such that $y \notin \beta(x, s_k)$. But we know $\{y\} \in \beta_{\mathbb{H}}(\{x\}, s_{k+2})$, so we have

$$y \in R_{\Sigma}(\{x\}, s_{k+2}) \subseteq R_{\beta}(\{x\}, s_k) = \beta(x, s_k)$$

which is the desired contradiction. Thus necessarily $s <_L r$ in $\mathbb{H}(X)$ as desired.

■

This lemma allows us to use the notation $<_L$ (or any of the other level relations) interchangeably for the space (X, R, β) or its hyperspace $(\mathbb{H}(X), R, \beta)$.

Definition 1.18 (Triangular) *A space is called triangular if for any $r, s \in R$ there exists a $t \in R$ such that, whenever $y \in \beta(x, r)$ and $z \in \beta(y, s)$, $z \in \beta(x, t)$. If t can be chosen such that $t =_L \max\{r, s\}$, then we call our space level-triangular.*

Lemma 1.19 *Let (X, R, β) be triangular, structured, and ordered, and let $A, B \subseteq X$ be totally r -bounded and totally s -bounded respectively. Then there exists a $t \in R$ such that $B \subseteq R_{\beta}(A, t)$.*

Proof:

Since A is totally r -bounded, there exist points $(x_i)_{i=1}^n$ such that $A \subseteq \bigcup_{i=1}^n \beta(x_i, r)$. Similarly there exist points $(y_i)_{i=1}^m$ such that $B \subseteq \bigcup_{i=1}^m \beta(y_i, s)$. Since our space is structured, for each i, j , there exists a $u_{ij} \in R$ such that $y_j \in \beta(x_i, u_{ij})$. Let $\bar{u} = \max\{u_{ij}\}$, and by triangularity, let $t \in R$ be such that if $y \in \beta(x, \bar{u})$ and $z \in \beta(y, s)$, then $z \in \beta(x, t)$. Now for any $z \in B$, $z \in \beta(y_j, s)$ for some j , and since $y_j \in \beta(x_i, \bar{u})$ for all i , it follows that $z \in \beta(x_i, t)$ for all i . Therefore, for all i ,

$$B \subseteq \beta(x_i, t) \subseteq R_{\beta}(A, t)$$

as desired. ■

Definition 1.20 (Proximity Set) A collection $\alpha \subseteq X$ is called a proximity set if

1. For all $x \in X$ and all $r \in R/\equiv_L$, there exists a point $\alpha_{x,r} \in L(x, r)$ such that $\alpha = \{\alpha_{x,r} : x \in X, r \in R/\equiv_L\}$
2. For any $x, y \in X$ and any $r, s \in R/\equiv_L$ with $s \leq_L r$, if $\alpha_{x,r} \in L(y, s)$, then $\alpha_{x,r} = \alpha_{y,s}$

Definition 1.21 (α -Close) Given a proximity set α , a set $A \subseteq X$ is called α -close if for all $r \in R$, there exists an $s <_L r$ such that $A \cap L(x, r) \subseteq \beta(\alpha_{x,r}, s)$ for all x .

Definition 1.22 (α -Hyperspace) Let (X, R, β) be a β -space, and let α be a proximity set. The α -hyperspace of X , denoted $\mathbb{H}_\alpha(X)$ (or generally just $\mathbb{H}(X)$), is defined to be

$$\mathbb{H}(X) = \{A \subseteq X : A \text{ is } \alpha\text{-close, radially complete, and totally } r\text{-bounded for some } r\}$$

The sets $A \in \mathbb{H}(X)$ are called α -hypersets, or hypersets.

Lemma 1.23 Let (X, R, β) be a β -space, and let $t_1, t_2, t_3 \in R$ with t_{i+1} a swing value for t_i . Then for any x , $\beta(x, t_3) \subseteq \beta(x, t_1)$.

Proof:

Pick any $y \in \overline{\beta(x, t_3)}$. Then for any $p \in R$, $\beta(y, p) \cap \beta(x, s) \neq \emptyset$. In particular, we can find some $\alpha \in \beta(y, t_3) \cap \beta(x, t_3)$. Then clearly

$$\beta(y, t_3), \beta(x, t_3) \subseteq \beta(\alpha, t_2) \subseteq \beta(x, t_1)$$

Since y was arbitrary, it follows that $\overline{\beta(x, t_3)} \subseteq \beta(x, t_1)$ as desired. \blacksquare

Proposition 1.24 Let (X, R, β) be ordered, swing-complete, level-countable, and level-triangular. Then for any choice of a proximity set α , the hyperspace $\mathbb{H}(X)$ is level-structured.

Proof:

Let $B \in L_\mathbb{H}(A, r)$, and let $(r_i)_{i=1}^\infty$ be a swing sequence for r such that $B \in \beta_\mathbb{H}(A, r_i)$ for all i . Since our space is swing-complete, without loss of generality we can take $(r_i)_{i=1}^\infty$ to be a level swing sequence. Now since A and B are both α -close, let $t_4 <_L r$ be such that

1. $A \cap L(x, r) \subseteq \overline{\beta(\alpha_{x,r}, t_4)}$
2. $B \cap L(x, r) \subseteq \overline{\beta(\alpha_{x,r}, t_4)}$

By level-triangularity, we can find $t_1, t_2, t_3 \in R$ such that $t_1, t_2, t_3 \equiv_L t_4$ and where t_{i+1} is a swing value of t_i . We will show that $B \in \beta_\mathbb{H}(A, t_1)$.

Pick any $x \in B$. For any i , since $B \in \beta_\mathbb{H}(A, r_i)$, $B \subseteq R_\Sigma(A, r_i)$, so that $B \subseteq R_\beta(A, r_{i-2})$ by Lemma 1.12. Therefore, we can find a $y_i \in A$ such that $x \in \beta(y_i, r_{i-2})$, and thus that $y_i \in \beta(x, r_{i-3})$. Now we claim

that $(y_i) \xrightarrow{r} x$. To see this, let $q =_L r$. Since our space is level-countable, there is some k such that $r_k \leq q$. But for all $n \geq k + 3$,

$$y_n \in \beta(x, r_{n-3}) \subseteq \beta(x, r_k) \subseteq \beta(x, q)$$

as desired. Now since A is radially complete, in particular A is r -complete, so that there is some $y \in A$ such that $(y_i) \xrightarrow{r} y$. Then by Proposition ??, $y \in L(x, r)$. But then both $x \in \beta(\alpha_{x,r}, t_4)$ and $y \in \beta(\alpha_{x,r}, t_4)$, so that by Lemma 1.23 we have $x, y \in \beta(\alpha_{x,r}, t_2)$, and thus $x \in \beta(y, t_1)$. Since x was arbitrary, it follows that

$$B \subseteq R_\beta(A, t_1) \subseteq R_\Sigma(A, t_1)$$

Now the symmetric argument establishes that $A \subseteq R_\Sigma(B, t_1)$, so that we have $B \in \beta_\mathbb{H}(A, t_1)$ as desired. Since our space is ordered, by taking the maximum of the associated radial values, this shows that such a $t <_L r$ exists for any finite collection $(B_i)_{i=1}^n \subseteq L_\mathbb{H}(A, r)$, and so the proof is concluded. ■

Theorem 1.25 (Hyperspace Inheritance) *Let (X, R, β) be ordered, roll-ordered, level-structured, level-countable, swing complete, radially complete, level-triangular, Hausdorff, and CDLB. Further let α be a proximity set on X . Then the hyperspace $(\mathbb{H}(X), R, \beta_\mathbb{H})$ inherits the properties of being ordered, level structured, radially complete, and CDLB.*

Proof:

1. **Ordered:** This was shown in Theorem 1.14.
2. **Level Structured:** Let $A, B_1, \dots, B_n \subseteq X$. Since our space is triangular, structured, and ordered, we may apply Lemma 1.19 to obtain $(t_i)_{i=1}^n \subseteq R$ such that

$$B_i \subseteq R_\beta(A, t_i) \subseteq R_\Sigma(A, t_i)$$

Letting $t = \max\{t_i : i = 1, 2, \dots, n\}$, clearly $B_i \subseteq R_\Sigma(A, t)$, so that $\mathbb{H}(X)$ is structured as desired. Since our space is additionally swing-complete, level-countable, and level-triangular, we may apply Proposition 1.24 to obtain that $\mathbb{H}(X)$ is level-structured.

3. **Radially Complete:** First we show the inheritance of being r -complete for any r . Let $(A_i)_{i=1}^\infty$ be r -Cauchy. Define

$$A = \{x : \exists (x_i)_{i=1}^\infty \xrightarrow{r} x \text{ and } x_i \in A_i\}$$

HERE, CITE A LEMMA PROVING THAT $A \in \mathbb{H}(X)$. Let $s =_L r$, and let t be a level swing value of s . Let N be such that, for all $n, m \geq N$, $A_n \in \beta_\mathbb{H}(A_m, t)$. Then $A_n \subseteq R_\Sigma(A_m, t)$ for all such n, m . Pick any $x \in A$. Then there is a sequence $(x_i)_{i=1}^\infty \xrightarrow{r} x$ with $x_i \in A_i$. Thus, for all $n, m \geq N$, $x_n \in R_\Sigma(A_m, t)$. But since $(x_i)_{i=1}^\infty \xrightarrow{r} x$ and $t =_L r$, there is an $N_2 \geq N$ such that, by Proposition ??, for all $n \geq N_2$ and all $m \geq N$,

$$x \in \beta(x_n, t) \subseteq R_\Sigma(x_n, t) \subseteq R_\Sigma(R_\Sigma(A_m, t), t) \subseteq R_\Sigma(A_m, s)$$

where the last inclusion follows by Lemma 1.11. Thus we have that $A \subseteq R_\Sigma(A_m, s)$ for all $m \geq N$.

Letting t , s , and N be as before, for any $n \geq N$, pick $y \in A_n$. Let $(t_i)_{i=1}^\infty$ be a level swing sequence for t . We now define a sequence $(y_i)_{i=1}^\infty$ as follows. For $i < n$, pick $y_i \in A_i$ arbitrarily. Set $y_n = y$. Now let $(k_i)_{i=1}^\infty$ be a strictly increasing sequence of integers such that $k_1 = N$ and $A_j \in \beta_\mathbb{H}(A_{k_i}, t_i)$ for all $j \geq k_i$. If y_{k_j} has been chosen and $k_j < k \leq k_{j+1}$, choose $y_k \in A_k$ with $y_k \in \beta(y_{k_j}, t_{j-3})$. To see that this is possible, we note that $A_{k_j} \subseteq R_\Sigma(A_k, t_j)$ implies that there is a $y_k \in A_k$ such that $y_{k_j} \in \beta(y_k, t_{j-2})$ by the Geometric Series Lemma. Therefore $y_k \in \beta(y_{k_j}, t_{j-3})$ as desired.

We now argue that $(y_i)_{i=1}^\infty$ is r -Cauchy. Pick any $u =_L r$. Then since our space is level-countable, there is a $p \in \mathbb{N}$ such that $t_p \leq u$. Let α be such that $k_\alpha \geq p$. Then $y_k \in \beta(y_{k_{\alpha+6}}, t_{\alpha+3})$ for $k_{\alpha+6} \leq k \leq k_{\alpha+7}$. By construction of the sequence, for any $k, j \geq k_{\alpha+6}$,

$$y_k \in R_\Sigma(\{y_{k_{\alpha+6}}\}, t_{\alpha+3}) \subseteq \beta(y_{k_{\alpha+6}}, t_{\alpha+1}) \subseteq \beta(y_j, t_\alpha) \subseteq \beta(y_j, u)$$

where the middle inclusions follow again by the Geometric Series Lemma and its corollary. Thus $(y_i)_{i=1}^\infty$ is r -Cauchy, and since our space is r -complete, there is an $x \in X$ such that $(y_i) \xrightarrow{r} x$.

4. **CDLB:** Let $(r_i)_{i=1}^\infty$ be such that $r_{i+1} <_L r_i$ in $\mathbb{H}(X)$, and let $s \in R$ be arbitrary. Then by Lemma 1.17, $r_{i+1} <_L r_i$ in X as well. Since X is CDLB, there is a k such that $r_k \leq_L s$ in X . Again by Lemma 1.17, this implies that $r_k \leq_L s$ in $\mathbb{H}(X)$.

■

Definition 1.26 If (X, R, β) is an ordered β -space, a function $s : R \rightarrow R$ is called *strict* if it is strictly monotonic and level-surjective; that is, if

1. If $r < t$ then $s(r) < s(t)$
2. For any $t \in R$, there is an $r \in R$ such that $s(r) =_L t$

Definition 1.27 (Strict Contraction) A function $f : X \rightarrow X$ is called a *strict contraction* if there exists a strict $s : R \rightarrow R$ and an $N \in \mathbb{N}$ such that

1. $s(r) \leq r$ for all r
2. $s^N(r)$ is a swing value for r for any r
3. $f(\beta(x, r)) \subseteq \beta(f(x), s(r))$ for any x, r

Given a proximity set α , we say that f is α -preserving if it also satisfies

4. For any $t \in R$ and $y \in X$, $f(L(x, r)) = L(y, t)$ for all $x \in f^{-1}(y)$ and all $r \in s^{-1}(\{t' =_L t\})$
5. $f(\alpha_{x,r}) \in L(\alpha_{f(x), s(r)}, s(r))$

Proposition 1.28 *Let (X, R, β) be ordered. Let α be a proximity set and let $f : X \rightarrow X$ be α -preserving. Then if $A \subseteq X$ is α -close, necessarily $f(A)$ is α -close.*

Proof:

For any $t \in R$, let $r \in R$ be such that $s(r) = t$ and $f(L(x), r) \subseteq L(f(x), t)$. Let $q <_L r$ be such that

$$A \cap L(x, r) \subseteq \beta(\alpha_{x,r}, q)$$

for all x . Then for any y ,

$$f(A) \cap L(y, t) = f(A \cap f^{-1}(L(y, t)))$$

Now if $f(x) \in L(y, t)$, then $f(L(x, r)) \subseteq L(f(x), t) = L(y, t)$, so that

$$f^{-1}(L(y, t)) = \bigcup_{f(x) \in L(y, t)} L(x, r)$$

Therefore we have

$$\begin{aligned} f(A) \cap L(y, t) &= f\left(A \cap \bigcup_{f(x) \in L(y, t)} L(x, r)\right) \\ &= f\left(\bigcup_{f(x) \in L(y, t)} A \cap L(x, r)\right) \\ &\subseteq f\left(\bigcup_{f(x) \in L(y, t)} \beta(\alpha_{x,r}, q)\right) \\ &= \bigcup_{f(x) \in L(y, t)} f(\beta(\alpha_{x,r}, q)) \\ &\subseteq \bigcup_{f(x) \in L(y, t)} \beta(\alpha_{f(x), s(r)}, s(q)) \\ &= \bigcup_{f(x) \in L(y, t)} \beta(\alpha_{y,t}, s(q)) \\ &= \beta(\alpha_{y,t}, s(q)) \end{aligned}$$

But since $q <_L r$ and s is level-strict monotonic, $s(q) <_L s(r) = t$, so that $f(A)$ is α -close as desired. ■

Proposition 1.29 *Let α be the standard proximity set for \mathcal{L}^n , and let $f : \mathcal{L}^n \rightarrow \mathcal{L}^n$ satisfy that XXXXXXXX*

2 Examples

All our examples will be generated in \mathcal{L}^n , which we know from our work above, together with some facts proven in the β -Spaces paper, that these

spaces are sufficiently nice so that we may generate an α -hyperspace, apply the Contraction Mapping Theorem, and so on. We first need to know which contractions we may use.

Lemma 2.1 *Any function $f : \mathcal{L}^n \rightarrow \mathcal{L}^n$ that is affine and contractive in each coordinate is α -preserving.*

Example 2.2 (Cantor Set) *Taking \mathcal{L} to be our space, let F be the IFS containing*

$$\begin{aligned} f_1(y) &= \frac{1}{3}y \\ f_2(y) &= \frac{1}{3}y + \frac{1}{3} \end{aligned}$$

Then the invariant set for F is precisely the set

$$\{g(x) = c : c \in C\}$$

where $C \subseteq \mathbb{R}$ denotes the Cantor Set. To see that this is the correct invariant set, it suffices to note that 1) this set is clearly fixed under iterations of F , and 2) this set is an α -hyperset.

Example 2.3 (Stretched Cantor Set) *Let F be the IFS containing*

$$\begin{aligned} f_1(y) &= \frac{1}{3}y \\ f_2(y) &= \frac{1}{3}y + \frac{2}{3x} \end{aligned}$$

where we recall that x is the variable in the functions in the field \mathcal{L} of formal Laurent polynomials over \mathbb{R} , and thus that $\frac{2}{3x}$ is the (infinite) element of \mathcal{L} given by $g(x) = \frac{2}{3x}$. As before we can see that the unique invariant α -hyperset is

$$C_{\text{stretched}} = \left\{ g(x) = \frac{c}{x} : c \in C \right\}$$

Example 2.4 (Small Cantor Set) *Let F be the IFS containing*

$$\begin{aligned} f_1(y) &= x \cdot y \\ f_2(y) &= x \cdot y + (1 - x) \end{aligned}$$

After some consideration, one can see that the invariant set here is

$$C_{\text{small}} = \left\{ g(x) : g(x) = \sum_{i=0}^k (-1)^i x^{f(i)} \text{ where } f(0) \geq 0, k \geq -1, f \text{ strictly monotonic} \right\}$$

Again, to see that this is the unique invariant α -hyperset for F , we simply note that this set is invariant under application of F , and that it is an α -hyperset.

The Cantor Set constructed in \mathcal{L} (obviously) and the Stretched Cantor Set (less obviously) are still “equivalent” to the Cantor Set as usually constructed as a subset of \mathbb{R} . In particular both of these fractals are metrizable. However, the Small Cantor Set C_{small} is not, and so provides our first real example of a non-metrizable fractal. To see that C_{small} is not metrizable, we simply note that C_{small} contains values on multiple levels; in particular, $x^n \in C_{\text{small}}$ for all $n \geq 0$.